Stochastic coherence resonance in a bistable system with fractional damping

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Abstract: A bistable dynamical system with the Duffing potential, fractional damping, and random excitation has been modelled. To excite the system, we used a stochastic force defined by Wiener random process of Gaussian distribution. As expected, stochastic resonance appeared for sufficiently high noise intensity. We estimated the critical value of the noise level as a function of derivative order q. For smaller order q, damping enhancement was reported.

1. Introduction

Fractional order systems have been intensively studied in various contexts [1]. In mechanical engineering, there were suggestions to apply it for various complex non-viscous, memory effected damping effect like rubbing or composite material response including natural wooden composites. Finally, it was used to model viscoelatic properties [2-5].

Particularly, the fractional derivative was used to characterize visco-elastic properties of beams and plates [1]. Important features of such systems include their dynamical memory of previous states, which could imply additional internal variables [6, 7]. Consequently, the modelling of the system instanenous states involves their time evolution history. Note that the memory effects combined with additional nonlinearities can be a source of hysteresis which is very common in engineering systems [1,8,9]. The fractional order damped rotor system with rubbing malfunction was proposed [10, 11].

On the other hand, randomly excited nonliner systems show a number of interesting features such as stochastic Hopf bifurcation [8], period-doubling bifurcations [9] and a stochastic resonance phenomenon [12]. This resonance is characterized by the flow over the potential barrier. One of the simplest system with such a barrier may be defined as a single degree-of-freedom, double well Duffing potential. In such a system, the occurrence of a single well escape is a result of competition between damping and excitation. Consequently, this escape can be associated with stochastic resonance (or coherence resonance [13]). This phenomenon is expected to be more complicated in higher system dimensions (or memory effect), which can be introduced by hidden variables of non-viscous damping [7]. Motivated by mechanical engineering applications, fractional damping effects were studied in the context of resonance conditions, synchronization effects, and also appearance of chaotic solutions [14, 15]. It was found out [16] that the existence of the fractional-order derivative could affect not only damping, but also stiffness, which were characterized by equivalent damping and equivalent stiffness coefficients, respectively. The fractional calculus is going to have a fruitful field in many scientific areas. Assumed in the present paper, the cubic term would be useful in modelling a realistic nonlinear response of spring and dry friction system as reported in paper [17]. The approach of the fractional calculus looks promising to model a nonlinear phenomenon of the dry friction model found in [17]. It would be adopt to description of diffusion and wave propagation phenomenon, the system identification in robotics, telecommunications, and also for control systems [18]. Recently, the phenomenon of vibrational resonance was also investigated in wide parameter range of Duffing systems with fractional-order damping [19]. The authors of Ref. [19] claimed that factional-order damping can cause a change in a number of the steady stable states and then lead to single- or double-well resonance behaviour.

Cao et al. [20] investigated the fractionally damped system response by using phase diagrams, bifurcation diagrams and Poincare maps in a wide range of the fractional order changes from 0.1 to 2.0. Their analysis results show that the fractional order damped Duffing system could be treated as a bifurcation parameter. By continuing these studies, Chen at al. [21] and Hu at al. [22] analyzed such a system with a bounded noise excitation term composed of harmonic excitation with an additional random phase. The authors investigated the appearance of bimodal amplitude through a corresponding probability density. This signalled the existence of a stochastic jump.

In this paper, we continue the investigations described in [21, 22]. However, in contrast to Ref. [22], we terminate the harmonic component and study the non–linear Duffing system with a fractional derivative subjected to a random excitation force defined as generated with an additive white Gaussian noise term.

2. The model and equations of motion

Our discussion starts with the corresponding Duffing equation supplemented by additional fractional damping and random forcing

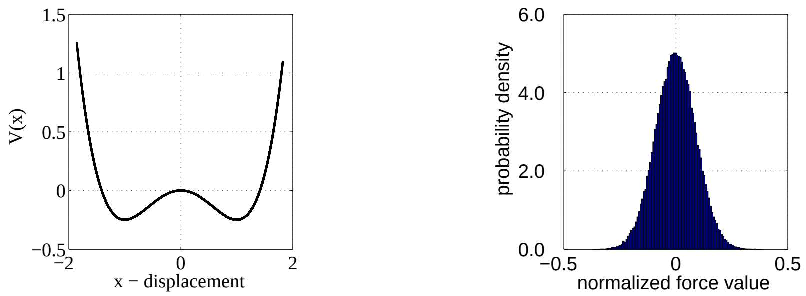
$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \beta \frac{\mathrm{d}^q x}{\mathrm{d}t^q} - x + x^3 = f(t), \tag{1}$$

where $\mathrm{d}^q x/\mathrm{d}t^q$ is the Grünwald-Letnikov fractional derivative [1, 22] with an order q

$$\frac{\mathrm{d}^q x}{\mathrm{d}t^q} =_a D^q x(t) = \lim_{\Delta t \to 0} \Big[\frac{1}{(\Delta t)^q} \sum_{j=0}^{\left[\frac{t-a}{\Delta t}\right]} (-1)^j \binom{q}{j} x(t - j\Delta t) \Big], \tag{2}$$

In the Eq. 2 $\left[\frac{t-a}{\Delta t}\right]$ means the integer part, where Δt denotes the integration time step, and a is an arbitrary number smaller than t. This defines the length of system memory. In the following analysis we assumed $a = 0$, which corresponds the memory length of whole trajectory. The binomial coefficients in the above sum can be expressed by the Euler's Gamma function

$$\binom{q}{j} = \frac{q!}{j!(q-j)!} = \frac{\Gamma(q+1)}{\Gamma(j+1)\Gamma(q-j+1)}, \tag{3}$$



(a) (b)

Figure 1. Bistable potential $V(x)$ (restore force (Eq. 1) $F_x = -\mathrm{d}V(x)/\mathrm{d}x = x - x^3$) used in the calculations (Eq. 5) (a), Gaussian probability distribution of a random force f(t). (Eq. 1) (b).

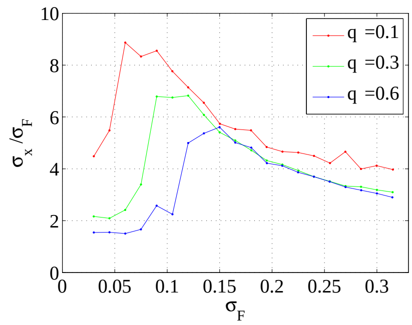


Figure 2. The displacement signals to noise ratio σ_x/σ_F versus noise intensity σ_F for different orders of derivative q.

Excitation force $f(t)$ has been defined as stationary Gaussian additive white noise with standard deviation σ_f described by the corresponding autocorrelation function:

$$\begin{aligned} < f(t)f(t+\Delta t') > &= 1/T_0 \int_0^{T_0} f(t)f(t+\Delta t')\mathrm{d}t \\ &= (\sigma_f)^2\delta(\Delta t'), \end{aligned} \quad (4)$$

where $\delta(.)$ is the Dirac delta, $\Delta t'$ is an arbitrary time difference, and T_0 is a large time interval.

The considered potential describes the restore force in Eq. 1

$$V(x) = \frac{x^2}{2} - \frac{x^4}{4} \quad (5)$$

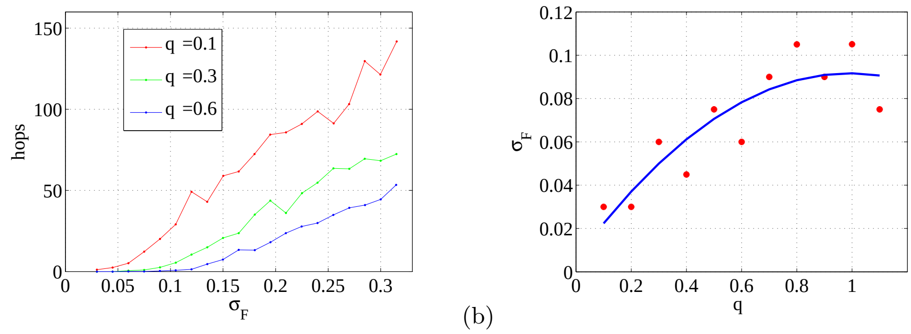
is given in Fig. 1a. Note that the dynamical model in Eq. 1 possesses damping and excitation terms. Due to competition between dissipating and generating mechanical energy, these two terms lead to balancing the total energy at a particular level. This level could be higher or lower with respect to the energy barrier ($\Delta V = 0.25$, see Fig. 1a). This level signals also a vicinity of cross-well jump conditions in the dynamical system.

Consequently, as level σ_f increases, the system response (measured by a standard deviation σ_x of the displacement x fluctuations) would be of sufficiently larger magnitude to pass the system states through the potential barrier.

Evidently, such a jump corresponds to the bifurcation of single potential well vibration of a relatively small amplitude into cross-barrier oscillations of a fairly large amplitude.

3. Simulation results

By simulating the dynamical system (Eq. 1) with increasing noise level conditions, we followed the scenario of stochastic coherence resonance [12, 22, 25]. For numerical calculations the Matlab environment has been used with non–dimensional parameters, where $\beta = 0.15$ is damping coefficient and excitation force represented by noise level is in range of $\sigma_F \in (0 \div 0.3)$. The initial conditions were fixed as $x_0 = 0.21, v_0 = 0.31$. The integration step $\Delta t = 0.005$, simulation time interval in terms of estimated time instants $t_n \in [0, 800]$, where first 400 instants were cut off as a transient part.



(a) (b)

Figure 3. Number of hops versus noise intensity σ_F for different orders of derivative q; and (b) critical curve of transitions begin between the potential wells via increasing orders of derivative q (red dots).

Figure 2 shows the signal-to-noise ratio versus increasing noise level σ_f. Note that the calculations have been made for different order of damping term q. For each simulation in terms of q, the averaged noise results have been plotted. The average include its 10 different Gaussian noise realizations. It is obvious that the stochastic resonance, corresponding to the increase of maximum σ_x/σ_F, is characterized by different noise level σ_f for different q.

On the way to a solution with the most frequent (coherent) jumps of the large amplitude, a single hop between potential wells occurs. Investigated this effect in greater detail in Fig. 3, we show the number of hops versus noise levels σ_f for various orders of derivative q. Figure 3b illustrates the dependence of the solution transition (single hop appearance between the potential wells in simulation time) via a function of critical noise level versus an increasing order of derivative q. The corresponding formula for plotting the curve presented in Fig. 3b can be expressed as a polynomial function, obtained in a standard approach by means of the least squares method:

$$\sigma_F(q) = -0.087q^2 + 0.17q + 0.0059. \tag{6}$$

4. Conclusions

In summary, our main results indicate that the decreasing order of derivative q (Eq. 1) enhances effective damping and leads to a different response of the system analyzed. Note that the present calculations were made for chosen system parameters but the final conclusion is of a general character. Namely, reaching cross barrier oscillations in a stochastic conditions, and simultaneous appearance of stochastic coherence resonance can be easier for smaller q (see Fig. 2). This implies smaller damping as a result of the fractional order.

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